1 Basés sur les polynômes de Lagrange

1.1 Généralités - Rappels

Le polynôme d'interpolation de Lagrange sur $(n+1)$ points $\{x_i, y_i = f(x_i)\}$, $i = 0, \ldots, n$ est donné par:

$$P_n(x) = \sum_{j=0}^{n} y_j L_j(x) \quad \text{, avec} \quad L_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

Dont la dérivée première est:

$$P_n^{(1)}(x) = \sum_{j=0}^{n} y_j L_j^{(1)}(x) \quad , \quad L_j^{(1)}(x) = \frac{\displaystyle\sum_{\substack{k=0 \\ k\neq j}}^{n} \left[\prod_{\substack{l=0 \\ l\neq k}}^{n} (x - x_l) \right]}{\displaystyle\prod_{\substack{k=0 \\ k\neq j}}^{n} (x_j - x_k)}$$

Et la dérivée seconde:

$$P_n^{(2)}(x) = \sum_{j=0}^{n} y_j L_j^{(2)}(x) \quad , \quad L_j^{(2)}(x) = \frac{\displaystyle\sum_{\substack{k=0 \\ k\neq j}}^{n} \left[\sum_{\substack{l=0 \\ l\neq k}}^{n} \left[\prod_{\substack{m=0 \\ m\neq l}}^{n} (x - x_m) \right] \right]}{\displaystyle\prod_{\substack{k=0 \\ k\neq j}}^{n} (x_j - x_k)}$$

etc...

En exprimant les valeurs des coefficients aux points de collocation (c.-à-d. lorsque $x = x_i$), on obtient les matrices de dérivation associées aux stencils de $(n+1)$ points. Le vecteur des dérivées d^{eme} $\vec{y}^{(d)}$ aux x_i est donné par $D_n^{(d)}.\vec{y}$, où $D_n^{(d)}$ est la matrice de d^{eme} dérivation associée au stencil de $(n+1)$ points.

1.2 Expressions des matrices de dérivation

Pour $n = 1$:

$$D_1^{(1)} = \begin{bmatrix} \frac{1}{x_0 - x_1} & \frac{1}{x_1 - x_0} \\ \frac{1}{x_0 - x_1} & \frac{1}{x_1 - x_0} \end{bmatrix}$$

Pour $n = 2$:

$$D_2^{(1)} = \begin{bmatrix} \frac{1}{x_0 - x_1} + \frac{1}{x_0 - x_2} & \frac{x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} & \frac{x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \\ \frac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} & \frac{1}{x_1 - x_0} + \frac{1}{x_1 - x_2} & \frac{x_1 - x_0}{(x_2 - x_0)(x_2 - x_1)} \\ \frac{x_2 - x_1}{(x_0 - x_1)(x_0 - x_2)} & \frac{x_2 - x_0}{(x_1 - x_0)(x_1 - x_2)} & \frac{1}{x_2 - x_0} + \frac{1}{x_2 - x_1} \end{bmatrix}$$

$$D_2^{(2)} = \begin{bmatrix} \frac{2}{(x_0 - x_1)(x_0 - x_2)} & \frac{2}{(x_1 - x_0)(x_1 - x_2)} & \frac{2}{(x_2 - x_0)(x_2 - x_1)} \\ \frac{2}{(x_0 - x_1)(x_0 - x_2)} & \frac{2}{(x_1 - x_0)(x_1 - x_2)} & \frac{2}{(x_2 - x_0)(x_2 - x_1)} \\ \frac{2}{(x_0 - x_1)(x_0 - x_2)} & \frac{2}{(x_1 - x_0)(x_1 - x_2)} & \frac{2}{(x_2 - x_0)(x_2 - x_1)} \end{bmatrix}$$

Pour $n = 3$:

$$D_3^{(1)} = \begin{bmatrix} \frac{1}{x_0-x_1}+\frac{1}{x_0-x_2}+\frac{1}{x_0-x_3} & \frac{(x_0-x_2)(x_0-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{(x_0-x_1)(x_0-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{(x_0-x_1)(x_0-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{(x_1-x_2)(x_1-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{1}{x_1-x_0}+\frac{1}{x_1-x_2}+\frac{1}{x_1-x_3} & \frac{(x_1-x_0)(x_1-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{(x_1-x_0)(x_1-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{(x_2-x_1)(x_2-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{(x_2-x_0)(x_2-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{1}{x_2-x_0}+\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3} & \frac{(x_2-x_0)(x_2-x_1)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{(x_3-x_1)(x_3-x_2)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{(x_3-x_0)(x_3-x_2)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{(x_3-x_0)(x_3-x_1)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{1}{x_3-x_0}+\frac{1}{x_3-x_1}+\frac{1}{x_3-x_2} \end{bmatrix}$$

$$D_3^{(2)} = \begin{bmatrix} \frac{2[(x_0-x_1)+(x_0-x_2)+(x_0-x_3)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{2[(x_0-x_2)+(x_0-x_3)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{2[(x_0-x_1)+(x_0-x_3)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{2[(x_0-x_1)+(x_0-x_2)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{2[(x_1-x_2)+(x_1-x_3)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{2[(x_1-x_0)+(x_1-x_2)+(x_1-x_3)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{2[(x_1-x_0)+(x_1-x_3)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{2[(x_1-x_0)+(x_1-x_2)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{2[(x_2-x_1)+(x_2-x_3)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{2[(x_2-x_0)+(x_2-x_3)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{2[(x_2-x_0)+(x_2-x_1)-(x_2-x_3)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{2[(x_2-x_0)+(x_2-x_1)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ \frac{2[(x_3-x_1)+(x_3-x_2)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} & \frac{2[(x_3-x_0)+(x_3-x_2)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} & \frac{2[(x_3-x_0)+(x_3-x_1)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} & \frac{2[(x_3-x_0)+(x_3-x_1)+(x_3-x_2)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{bmatrix}$$

Pour $n = 4$

$$D_4^{(1)} = \begin{bmatrix} \frac{1}{x_0-x_1}+\frac{1}{x_0-x_2}+\frac{1}{x_0-x_3}+\frac{1}{x_0-x_4} & \frac{(x_0-x_2)(x_0-x_3)(x_0-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} & \frac{(x_0-x_1)(x_0-x_3)(x_0-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} & \frac{(x_0-x_1)(x_0-x_2)(x_0-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} & \frac{(x_0-x_1)(x_0-x_2)(x_0-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ \frac{(x_1-x_2)(x_1-x_3)(x_1-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} & \frac{1}{x_1-x_0}+\frac{1}{x_1-x_2}+\frac{1}{x_1-x_3}+\frac{1}{x_1-x_4} & \frac{(x_1-x_0)(x_1-x_3)(x_1-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} & \frac{(x_1-x_0)(x_1-x_2)(x_1-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} & \frac{(x_1-x_0)(x_1-x_2)(x_1-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ \frac{(x_2-x_1)(x_2-x_3)(x_2-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} & \frac{(x_2-x_0)(x_2-x_3)(x_2-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} & \frac{1}{x_2-x_0}+\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}+\frac{1}{x_2-x_4} & \frac{(x_2-x_0)(x_2-x_1)(x_2-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} & \frac{(x_2-x_0)(x_2-x_1)(x_2-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ \frac{(x_3-x_1)(x_3-x_2)(x_3-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} & \frac{(x_3-x_0)(x_3-x_2)(x_3-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} & \frac{(x_3-x_0)(x_3-x_1)(x_3-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} & \frac{1}{x_3-x_0}+\frac{1}{x_3-x_1}+\frac{1}{x_3-x_2}+\frac{1}{x_3-x_4} & \frac{(x_3-x_0)(x_3-x_1)(x_3-x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ \frac{(x_4-x_1)(x_4-x_2)(x_4-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} & \frac{(x_4-x_0)(x_4-x_2)(x_4-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} & \frac{(x_4-x_0)(x_4-x_1)(x_4-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} & \frac{(x_4-x_0)(x_4-x_1)(x_4-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} & \frac{1}{x_4-x_0}+\frac{1}{x_4-x_1}+\frac{1}{x_4-x_2}+\frac{1}{x_4-x_3} \end{bmatrix}$$

$$
D_4^{(2)} = \begin{bmatrix}
\frac{2}{x_0-x_1}\left[\frac{1}{x_0-x_2}+\frac{1}{x_0-x_3}+\frac{1}{x_0-x_4}\right]+\frac{2}{x_0-x_2}\left[\frac{1}{x_0-x_3}+\frac{1}{x_0-x_4}\right]+\frac{2}{(x_0-x_3)(x_0-x_4)} \\
\frac{2\left[(x_1-x_2)[(x_1-x_3)+(x_1-x_4)]+(x_1-x_3)(x_1-x_4)\right]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
\frac{2\left[(x_2-x_1)[(x_2-x_3)+(x_2-x_4)]+(x_2-x_3)(x_2-x_4)\right]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
\frac{2\left[(x_3-x_1)[(x_3-x_2)+(x_3-x_4)]+(x_3-x_2)(x_3-x_4)\right]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
\frac{2\left[(x_4-x_1)[(x_4-x_2)+(x_4-x_3)]+(x_4-x_2)(x_4-x_3)\right]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
\\
\frac{2\left[(x_0-x_2)[(x_0-x_3)+(x_0-x_4)]+(x_0-x_3)(x_0-x_4)\right]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
\frac{2}{x_1-x_0}\left[\frac{1}{x_1-x_2}+\frac{1}{x_1-x_3}+\frac{1}{x_1-x_4}\right]+\frac{2}{x_1-x_2}\left[\frac{1}{x_1-x_3}+\frac{1}{x_1-x_4}\right]+\frac{2}{(x_1-x_3)(x_1-x_4)} \\
\frac{2\left[(x_2-x_0)[(x_2-x_3)+(x_2-x_4)]+(x_2-x_3)(x_2-x_4)\right]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
\frac{2\left[(x_3-x_0)[(x_3-x_2)+(x_3-x_4)]+(x_3-x_2)(x_3-x_4)\right]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
\frac{2\left[(x_4-x_0)[(x_4-x_2)+(x_4-x_3)]+(x_4-x_2)(x_4-x_3)\right]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
\\
\frac{2\left[(x_0-x_1)[(x_0-x_3)+(x_0-x_4)]+(x_0-x_3)(x_0-x_4)\right]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
\frac{2\left[(x_1-x_0)[(x_1-x_3)+(x_1-x_4)]+(x_1-x_3)(x_1-x_4)\right]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
\frac{2}{x_2-x_0}\left[\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}+\frac{1}{x_2-x_4}\right]+\frac{2}{x_2-x_1}\left[\frac{1}{x_2-x_3}+\frac{1}{x_2-x_4}\right]+\frac{2}{(x_2-x_3)(x_2-x_4)} \\
\frac{2\left[(x_3-x_0)[(x_3-x_1)+(x_3-x_4)]+(x_3-x_1)(x_3-x_4)\right]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
\frac{2\left[(x_4-x_0)[(x_4-x_1)+(x_4-x_3)]+(x_4-x_1)(x_4-x_3)\right]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
\\
\frac{2\left[(x_0-x_1)[(x_0-x_2)+(x_0-x_4)]+(x_0-x_2)(x_0-x_4)\right]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
\frac{2\left[(x_1-x_0)[(x_1-x_2)+(x_1-x_4)]+(x_1-x_2)(x_1-x_4)\right]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
\frac{2\left[(x_2-x_0)[(x_2-x_1)+(x_2-x_4)]+(x_2-x_1)(x_2-x_4)\right]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
\frac{2}{x_3-x_0}\left[\frac{1}{x_3-x_1}+\frac{1}{x_3-x_2}+\frac{1}{x_3-x_4}\right]+\frac{2}{x_3-x_1}\left[\frac{1}{x_3-x_2}+\frac{1}{x_3-x_4}\right]+\frac{2}{(x_3-x_2)(x_3-x_4)} \\
\frac{2\left[(x_4-x_0)[(x_4-x_1)+(x_4-x_2)]+(x_4-x_1)(x_4-x_2)\right]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
\\
\frac{2\left[(x_0-x_1)[(x_0-x_2)+(x_0-x_3)]+(x_0-x_2)(x_0-x_3)\right]}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
\frac{2\left[(x_1-x_0)[(x_1-x_2)+(x_1-x_3)]+(x_1-x_2)(x_1-x_3)\right]}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
\frac{2\left[(x_2-x_0)[(x_2-x_1)+(x_2-x_3)]+(x_2-x_1)(x_2-x_3)\right]}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
\frac{(x_3-x_0)(x_3-x_1)(x_3-x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
\frac{2}{x_4-x_0}\left[\frac{1}{x_4-x_1}+\frac{1}{x_4-x_2}+\frac{1}{x_4-x_3}\right]+\frac{2}{x_4-x_1}\left[\frac{1}{x_4-x_2}+\frac{1}{x_4-x_3}\right]+\frac{2}{(x_4-x_2)(x_4-x_3)}
\end{bmatrix}
$$

1.3 Expressions des matrices de dérivation sur des points équidistants

Lorsque $x_i = ih$, $i = 0, \ldots, n$:

Pour $n = 1$:

$$D_1^{(1)} = \frac{1}{h} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Pour $n = 2$:

$$D_2^{(1)} = \frac{1}{2h} \begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix} \qquad D_2^{(2)} = \frac{1}{h^2} \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Pour $n = 3$:

$$D_3^{(1)} = \frac{1}{6h} \begin{bmatrix} -11 & 18 & -9 & 2 \\ -2 & -3 & 6 & -1 \\ 1 & -6 & 3 & 2 \\ -2 & 9 & -18 & 11 \end{bmatrix} \qquad D_3^{(2)} = \frac{1}{h^2} \begin{bmatrix} 2 & -5 & 4 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 4 & -5 & 2 \end{bmatrix}$$

Pour $n = 4$:

$$D_4^{(1)} = \frac{1}{12h} \begin{bmatrix} -25 & 48 & -36 & 16 & -3 \\ -3 & -10 & 18 & -6 & 1 \\ 1 & -8 & 0 & 8 & -1 \\ -1 & 6 & -18 & 10 & 3 \\ 3 & -16 & 36 & -48 & 25 \end{bmatrix} \qquad D_4^{(2)} = \frac{1}{12h^2} \begin{bmatrix} 35 & -104 & 114 & -56 & 11 \\ 11 & -20 & 6 & 4 & -1 \\ -1 & 16 & -30 & 16 & -1 \\ -1 & 4 & 6 & -20 & 11 \\ 11 & -56 & 114 & -104 & 35 \end{bmatrix}$$